Two kinds of generalized Taub-NUT metrics and the symmetry of associated dynamical systems

1994 J. Phys. A: Math. Gen. 27 3179

(http://iopscience.iop.org/0305-4470/27/9/029)

J. Phys. A: Math. Gen. 27 (1994) 3179–3190. Printed in the UK

Two kinds of generalized Taub–NUT metrics and the symmetry of associated dynamical systems

Toshihiro Iwai† and Noriaki Katayama‡

† Department of Applied Mathematics and Physics, Kyoto University, Kyoto 606-01, Japan
‡ Department of Systems and Control Engineering, Osaka Prefectural College of Technology, Neyagawa, Osaka 572, Japan

Received 4 November 1993

Abstract. A number of researches have been made for the (Euclidean) Taub–NUT metric, because the geodesic for this metric describes approximately the motion of well separated monopole–monopole interaction. From the viewpoint of dynamical systems, it is well known also that the Taub–NUT metric admits the Kepler-type symmetry, and hence provides a non-trivial generalization of the Kepler problem. More specifically speaking, because of an $U(1)$ symmetry, the geodesic flow system as a Hamiltonian system for the Taub–NUT metric is reduced to a Hamiltonian system which admits a conserved Runge–Lenz-like vector in addition to the angular momentum vector, and thereby whose trajectories turn out to be conic sections. In particular, all the bounded trajectories of the reduced system are closed. In this paper, the Taub–NUT metrics is generalized so that the reduced system may remain to have the property that all of bounded trajectories are closed. On the application of Bertrand's method to the reduced system, two types of systems are found: one is called the Kepler-type system and the other the harmonic oscillator-type system. Correspondingly, two types of metrics come out: the Kepler-type metric and the harmonic oscillator-type metric. Furthermore, the symmetry of the Kepler-type system and of the harmonic oscillator-type system are studied through forming accidental first integrals. Thus the generalization of the Taub–NUT metric accomplishes non-trivial generalizations of the Kepler problem and the harmonic oscillator.

1. Introduction

It is well known in classical mechanics that only the Kepler problem and the harmonic oscillator are those central-potential dynamical systems whose bounded trajectories are all closed. This fact was proved by Bertrand (1873) in the last century, and is referred to as Bertrand's theorem. Furthermore, these systems have been well studied for a long time for their remarkable symmetry.

On the other hand, active attention has been paid to the Taub–NUT metric, because the motion of well separated monopole–monopole interactions is described approximately by the geodesics of the Taub–NUT metric (Manton 1982, 1985, Atiyah and Hitchin 1985). From the viewpoint of dynamical systems, the geodesic motion of the Taub–NUT metric is known to admit the Kepler-type symmetry (Gibbons and Manton 1986, Gibbons and Ruback 1987, 1988, Fehér and Horváthy 1987, Cordani, Fehér, and Horváthy 1988, 1990). To be precise, as a Hamiltonian system, the geodesic flow system for the Taub–NUT metric is reduced to a Hamiltonian system on $T^*(\mathbb{R}^3 - \{0\})$, which admits the Kepler-type symmetry. One can actually find the so-called Runge–Lenz vector in addition to the angular momentum vector. As a consequence, all of the bounded trajectories are closed, and those conserved vectors put together are shown to be closed under the Poisson brackets with respect to the reduced symplectic form on the reduced phase space $T^*(\mathbb{R}^3 - \{0\})$. Thus the Taub–NUT metric provides a non-trivial generalization of the Kepler problem.

In view of Bertrand's theorem, a question arises as to what metric will provide a non-trivial generalization of the harmonic oscillator. In clearing up this question, Bertrand's method will provide a guiding principle. The main interest of this paper is accordingly the periodicity of trajectories of a reduced system defined on $T^*(\mathbb{R}^3 - \{0\})$. One may look forward to finding a metric which admits the harmonic oscillator-type symmetry in contrast to the Taub–NUT metric which admits the Kepler-type symmetry. To this end, a generalized Taub–NUT metric with undetermined functions, $f(r)$ and $g(r)$, of radius r is to be defined on $\mathbb{R}^4 - \{0\}$. The geodesic flow system for this metric is a Hamiltonian system on $T^*(\mathbb{R}^4 - \{0\})$ and can be reduced to a Hamiltonian system on $T^*(\mathbb{R}^3 - \{0\})$ by using the $U(1)$ symmetry. By the application of Bertrand's method, the functions $f(r)$ and $g(r)$ are to be determined so that bounded trajectories of the reduced system may be all closed, and thereby two types of reduced systems are specified. One is called the Kepler-type system, and the other the harmonic oscillator-type system, which are named after the form of the Hamiltonian with specified functions $f(r)$ and $g(r)$. Correspondingly, two types of generalized Taub–NUT metrics are found, the Kepler-type metric and the harmonic oscillator-type metric. Thus the non-trivial generalization of the harmonic oscillator is accomplished. In spite of being old-fashioned, the Bertrand method proves to be of practical use in finding dynamical systems of marked periodicity property. The Kepler-type metric was already found in another method and investigated in a preceding paper (Iwai and Katayama 1993) in the name of the extended Taub–NUT metric; according to this paper, the extended Taub–NUT metric comes to have a self-dual Riemann curvature tensor or a conformally self-dual Weyl curvature tensor, depending on the choice of parameters included. It was also shown in the same paper that the Kepler-type system admits Kepler-type symmetry.

The plan of this paper is as follows. Section 2 contains the setting up for the reduced Hamiltonian system from the geodesic flow system for a generalized Taub–NUT metric. In section 3, a necessary and sufficient condition will be found for any bounded trajectories of the reduced Hamiltonian system to be closed, after the Bertrand method. Two types of Hamiltonian systems, the Kepler-type system and the harmonic oscillator-type system, will come out and thereby two types of generalized Taub–NUT metrics will be determined, the Kepler-type metric and the harmonic oscillator-type metric. Section 4 is concerned with the symmetry of the reduced Hamiltonian systems obtained above. The symmetry to be found will make sure that the Kepler-type and the harmonic oscillator-type systems are indeed a non-trivial generalization of the the Kepler problem and of the harmonic oscillator, respectively. The Kepler-type system in this article is also an extension of the Hamiltonian system which is reduced from the geodesic flow system for the Taub–NUT metric. As for a generalization of the harmonic oscillator, over twenty years ago McIntosh and Cisneros (1970) considered the harmonic oscillator in Dirac's monopole field, but they were unable to succeed in discussing the symmetry of that system. The harmonic oscillator-type system found in this article is a generalization of their system as well, and is shown to admit the harmonic oscillator-type symmetry, which this system is expected to have. Section 5 contains remarks.

2. Setting up

Let (y_ν), $\nu = 1, \cdots, 4$, be the Cartesian coordinates of $\mathbb{R}^4$. We introduce the curvilinear coordinates (r, θ, ϕ, ψ) by

$$
\begin{aligned}
y_1 &= \sqrt{r}\cos\frac{\theta}{2}\cos\frac{\psi+\phi}{2} & \qquad y_2 &= \sqrt{r}\cos\frac{\theta}{2}\sin\frac{\psi+\phi}{2} \\
y_3 &= \sqrt{r}\sin\frac{\theta}{2}\cos\frac{\psi-\phi}{2} & \qquad y_4 &= \sqrt{r}\sin\frac{\theta}{2}\sin\frac{\psi-\phi}{2}
\end{aligned}
\tag{2.1}
$$

where $r > 0$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$, $0 \leqslant \psi \leqslant 4\pi$. We are going to consider a generalized Taub–NUT metric in the form

$$
\mathrm{d}s_G^2 = f(r)(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)) + g(r)(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)^2 \tag{2.2}
$$

where $f(r)$ and $g(r)$ are arbitrary functions of r. For $f(r) = 1 + (4m/r)$ and $g(r) = (4m)^2/1 + (4m/r)$, $\mathrm{d}s_G^2$ becomes the Euclidean Taub–NUT metric.

The Lagrangian of the geodesic flow on the tangent bundle $T(\mathbb{R}^4 - \{0\})$ takes the form

$$
L = \tfrac{1}{2}f(r)(\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)) + \tfrac{1}{2}g(r)(\dot{\psi} + \cos\theta\,\dot{\phi})^2 \tag{2.3}
$$

where $(r, \theta, \phi, \psi, \dot{r}, \dot{\theta}, \dot{\phi}, \dot{\psi})$ are local coordinates of the tangent bundle $T(\mathbb{R}^4 - \{0\})$. Since ψ is a cyclic variable,

$$
\mu = g(r)(\dot{\psi} + \cos\theta\,\dot{\phi}) \tag{2.4}
$$

is a conserved quantity. This fact is in keeping with the bundle structure $\mathbb{R}^4 - \{0\} \to \mathbb{R}^3 - \{0\}$ with structure group $U(1)$, whose action is expressed as $\psi \to \psi + t$. In the Hamiltonian formalism, the conserved quantity is obtained as follows: The infinitesimal generator of the $U(1)$ action takes the form $\partial/\partial\psi$, so that the conserved momentum is given by $\mu = p_\psi = \gamma(\partial/\partial\psi)$, where γ is the canonical one-form on the cotangent bundle $T^*(\mathbb{R}^4 - \{0\})$ with local coordinates $(r, \theta, \phi, \psi, p_r, p_\theta, p_\phi, p_\psi)$.

The reduction procedure with $\mu \in \mathbb{R}$ fixed results in the Hamiltonian system on $T^*(\mathbb{R}^3 - \{0\}) \cong (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$ with Cartesian coordinates (x_j, p_j), $j = 1, 2, 3$,

$$
\omega_\mu = \sum_{j=1}^{3} \mathrm{d}p_j \wedge \mathrm{d}x_j - \frac{\mu}{2r^3}\sum_{j,k,\ell} \varepsilon_{jk\ell} x_j\,\mathrm{d}x_k \wedge \mathrm{d}x_\ell \tag{2.5}
$$

$$
H_\mu = \frac{1}{2f(r)}\sum_{j=1}^{3} p_j^2 + \frac{\mu^2}{2g(r)} \tag{2.6}
$$

where $r = \sqrt{\sum_{j=1}^3 x_j^2}$. For the reduction, see, for example, Iwai and Uwano (1986). We notice here that the variables (r, θ, ϕ) are viewed as the usual spherical coordinates in the space $\mathbb{R}^3 - \{0\}$.

We are to write out the equation of motion for the reduced system $(T^*(\mathbb{R}^3 - \{0\}), \omega_\mu, H_\mu)$. Let X_{H_μ} be the Hamiltonian vector field for H_μ, which is defined through $-\mathrm{d}H_\mu = \iota(X_{H_\mu})\omega_\mu$, where ι denotes the interior product. A calculation gives the X_{H_μ}, and hence the equation of motion in the form

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} &= \frac{\boldsymbol{p}}{f(r)} \\
\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= \left(\frac{f'(r)|\boldsymbol{p}|^2}{2rf(r)^2} + \mu^2\frac{g'(r)}{2rg(r)^2}\right)\boldsymbol{x} - \frac{\mu}{r^3 f(r)}\boldsymbol{p} \times \boldsymbol{x}
\end{aligned}
\tag{2.7}
$$

where $\times$ denotes the vector product operation and the prime means a derivative with respect to r.

Like the ordinary central-potential system, our system admits the angular momentum J due to manifest spherical symmetry;

$$J = x \times p + \frac{\mu}{r} x. \tag{2.8}$$

The fact that the inner product of J with x/r is a constant,

$$J \cdot \frac{x}{r} = \mu \tag{2.9}$$

implies that the trajectories of our system in the x-space lie on the cone whose axis is in the direction of J. Thus our system can be reduced further to a system of two degrees of freedom. To make things precise, we have to assume that $|J| \neq |\mu|$. In fact, from the definition (2.8), the square of J is given by

$$|J|^2 = |x \times p|^2 + \mu^2 \tag{2.10}$$

so that if $|J| = |\mu|$ the cone on which trajectories lie contracts to the axis in the direction of J.

The Hamiltonian system $(T^*(\mathbb{R}^3 - \{0\}), \omega_\mu, H_\mu)$ covers a class of dynamical systems of particular interest. For example, for

$$f(r) = 1 \qquad g(r) = \frac{r^2}{(1-r)^2} \tag{2.11}$$

the Hamiltonian H_μ has the form

$$H_\mu = \frac{1}{2}\sum_j p_j^2 + \frac{\mu^2}{2}\left(1 - \frac{1}{r}\right)^2 \tag{2.12}$$

which is called the MIC–Zwanziger system (Zwanziger 1968, McIntosh and Cisneros 1970). If one sets

$$f(r) = 1 \qquad g(r) = \frac{r^2}{1 - 2kr/\mu^2} \qquad k = \text{const} > 0. \tag{2.13}$$

one has the MIC–Kepler problem (McIntosh and Cisneros 1970, Iwai and Uwano 1986);

$$H_\mu = \frac{1}{2}\sum_j p_j^2 + \frac{\mu^2}{2r^2} - \frac{k}{r}. \tag{2.14}$$

As was touched upon already, for the functions

$$f(r) = \frac{4m + r}{r} \qquad g(r) = \frac{(4m)^2 r}{4m + r} \tag{2.15}$$

we have the reduced system from the geodesic flow system for the Euclidean Taub–NUT metric. The reduced Hamiltonian then takes the form

$$H_\mu = \frac{r}{4m + r}\left(\frac{1}{2}\sum_j p_j^2 + \frac{\mu^2}{32m^2}\left(1 + \frac{4m}{r}\right)^2\right). \tag{2.16}$$

These systems have the respective Runge–Lenz-like vectors in addition to the angular momentum J, and hence the trajectories in the x-space are shown to be conic sections (Gibbons and Manton 1986, Gibbons and Ruback 1987, 1988, Fehér and Horváthy 1987, Cordani, Fehér, and Horváthy 1988, 1990, Iwai and Uwano 1986). This means that all the bounded trajectories are closed.

Another example is obtained by setting

$$f(r) = 1 \qquad g(r) = \frac{r^2}{1 + kr^4/\mu^2} \qquad k = \text{const} > 0 \tag{2.17}$$

with a resulting Hamiltonian

$$H_\mu = \frac{1}{2}\sum_j p_j^2 + \frac{\mu^2}{2r^2} + \frac{1}{2}kr^2. \tag{2.18}$$

This system was treated by McIntosh and Cisneros (1970) and may be called the MIC-harmonic oscillator. All orbits for the MIC-harmonic oscillator are shown to be closed, which will be explained in the closing remarks of the next section.

In view of the above examples, we are prompted to ask when (or for what $f(r)$ and $g(r)$) all of the bounded trajectories of the Hamiltonian system $(T^*(\mathbb{R}^3 - \{0\}), \omega_\mu, H_\mu)$ are closed. This question will be worked out in the following section.

3. The Kepler-type and harmonic oscillator-type metrics

In this section, we are to derive a certain class of generalized Taub–NUT metrics. In the last section, we have derived the equation of motion for the reduced Hamiltonian system $(T^*(\mathbb{R}^3 - \{0\}), \omega_\mu, H_\mu)$. Further, we have observed from (2.9) that our system can be further reduced to a system of two degrees of freedom. Bertrand's method is applicable to this two-degrees-of-freedom system. We will then be able to find a necessary and sufficient condition for any bounded trajectories of the reduced system to be closed, and thereby to determine the unknown functions $f(r)$ and $g(r)$ in the Hamiltonian H_μ. Getting back to ds_G^2, these functions will provide the Kepler-type and the harmonic oscillator-type metrics as generalized Taub–NUT metrics.

Without loss of generality, we can set J in the direction of the x_3-axis. Then the variable θ is constant during the motion, so that the angular momentum vector, being associated with the infinitesimal rotation about the x_3-axis, is shown to satisfy

$$J := |J| = f(r)r^2\dot{\phi}. \tag{3.1}$$

From (2.3) the conserved energy can be put in the form

$$E = \tfrac{1}{2}f(r)\dot{r}^2 + \frac{J^2 - \mu^2}{2r^2 f(r)} + \frac{\mu^2}{2g(r)}. \tag{3.2}$$

Equations (3.1) and (3.2) provide the equations of motion on the cone. On introducing the variable $u = 1/r$, equations (3.1) and (3.2) are expressed, respectively, as

$$\frac{d\phi}{dt} = \frac{u^2 J}{f_1(u)} \tag{3.3}$$

$$\left(\frac{du}{dt}\right)^2 = \frac{2Eu^4}{f_1(u)} - \frac{u^6 J^2}{f_1(u)^2} + \frac{\mu^2 u^6}{f_1(u)^2} - \frac{\mu^2 u^4 g_1(u)}{f_1(u)^2} \tag{3.4}$$

where

$$f_1(u) = f(1/u) \qquad g_1(u) = \frac{f(1/u)}{g(1/u)}. \tag{3.5}$$

By taking ϕ as a parameter describing trajectories, we put the above equations together to obtain

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{J^2}\left(2Ef_1(u) - J^2u^2 + \mu^2u^2 - \mu^2 g_1(u)\right). \tag{3.6}$$

As pointed out in the last section, we have assumed that $|J| \neq |\mu|$, so that we are allowed to take (u, ϕ) as the coordinates of the cone. The case of $|J| = |\mu|$ will be dealt with at the end of this section. Equation (3.6) is the fundamental equation for which we are to investigate whether bounded trajectories are all closed. If the trajectory $u = u(\phi)$ is closed, u should have a maximum and a minimum. Let u_1 be a minimum and u_2 the following maximum the trajectory takes as ϕ increases. Since $du/d\phi = 0$ for those values, one has, from (3.6),

$$2Ef_1(u_k) - u_k^2 J^2 + \mu^2(u_k^2 - g_1(u_k)) = 0 \qquad k = 1, 2. \tag{3.7}$$

The increment of the angle, $\Delta\phi$, during the motion from $u = u_1$ to the following $u = u_2$ follows from (3.6) by

$$\Delta\phi = \int_{u_1}^{u_2} \frac{J\,\mathrm{d}u}{\sqrt{2Ef_1(u) - J^2u^2 + \mu^2u^2 - \mu^2 g_1(u)}}. \tag{3.8}$$

We assume further that no critical values of u exist between u_1 and u_2. Then a necessary and sufficient condition for the trajectory to be closed is that $\Delta\phi = q\pi$ for some rational number q. We now introduce functions $f_2(u)$ and $g_2(u)$ defined by

$$f_2(u) = f_1(u) - 1 \qquad g_2(u) = g_1(u) - u^2 \tag{3.9}$$

and, further, the function $V(u)$ by

$$V(u) = \tfrac{1}{2}\mu^2 g_2(u) - f_2(u)E \tag{3.10}$$

where E is considered as a constant. Then equation (3.8) turns into

$$\Delta\phi = \int_{u_1}^{u_2} \frac{J\,\mathrm{d}u}{\sqrt{2(E - V(u)) - J^2u^2}}. \tag{3.11}$$

This equation is the same as the one which Bertrand (1873) treated in the ordinary central-potential problem (see also Greenberg (1966)). For $V(u)$, the condition (3.7) is put in the form

$$2(E - V(u_k)) - J^2u_k^2 = 0 \qquad k = 1, 2. \tag{3.12}$$

According to Bertrand (1873), the condition $\Delta\phi = q\pi$ for any bounded trajectory to be closed determines the function $V(u)$ in the form

$$V(u) = \xi_0 u + \xi_1 \tag{3.13}$$

$$V(u) = \xi_0 u^{-2} + \xi_1 \tag{3.14}$$

where ξ_0 and ξ_1 are constants. Going backward, we will find the functions $f(r)$ and $g(r)$ from (3.13) and (3.14).

From (3.13) together with (3.9) and (3.10), one has

$$\tfrac{1}{2}\mu^2 g_2(u) - f_2(u)E = \xi_0 u + \xi_1. \tag{3.15}$$

Since μ and E should be arbitrary constants, both of $f_2(u)$ and $g_2(u)$ in the left-hand side of (3.15) must be inhomogeneously linear in u,

$$f_2(u) = \eta_1 u + \eta_2 \qquad g_2(u) = \eta_3 u + \eta_4 \tag{3.16}$$

where η_ν, $\nu = 1, \cdot, 4$, are constants. Then, through (3.5) and (3.9), $f(r)$ and $g(r)$ are found to be

$$f(r) = \frac{a + br}{r} \qquad g(r) = \frac{(a + br)r}{1 + cr + dr^2} \tag{3.17}$$

where a, b, c, d are constants. Conversely, the increment (3.8) for the functions given in (3.17) is expressed and calculated as

$$\Delta\phi = \int_{u_1}^{u_2} \frac{J\,\mathrm{d}u}{\sqrt{2bE - d\mu^2 + (2aE - c\mu^2)u - J^2u^2}} = \pi \tag{3.18}$$

if the quadratic polynomial in the radical has the positive roots, u_1 and u_2. Hence, any bounded trajectory with $J \neq \mu$ is closed. For the functions (3.17), the Hamiltonian (2.6) is put in the form

$$H_\mu = \frac{r}{a+br}\left(\frac{1}{2}\sum p_j^2 + \frac{\mu^2}{2r^2} + \frac{c\mu^2}{2r} + \frac{d\mu^2}{2}\right). \tag{3.19}$$

Since this Hamiltonian covers the MIC–Zwazinger- and the MIC–Kepler-problem Hamiltonians, (2.12) and (2.14), we call the Hamiltonian system $(T^*(\mathbb{R}^3 - \{0\}), \omega_\mu, H_\mu)$ with (3.19) the Kepler-type system. On a similar idea, we call the metric (2.2) together with (3.17) the Kepler-type metric, and denote it by $\mathrm{d}s_{\mathrm{K}}^2$;

$$\mathrm{d}s_{\mathrm{K}}^2 = \frac{a+br}{r}(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)) + \frac{(a+br)r}{1+cr+dr^2}(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)^2. \tag{3.20}$$

On the other hand, from (3.14) we are led to

$$f(r) = ar^2 + b \qquad g(r) = \frac{(ar^2+b)r^2}{1+cr^2+dr^4}. \tag{3.21}$$

Conversely, for these functions, the increment $\Delta\phi$ takes the value

$$\Delta\phi = \int_{u_1}^{u_2} \frac{J\,\mathrm{d}u}{\sqrt{2bE - c\mu^2 + (2aE - \mathrm{d}\mu^2)u^{-2} - J^2u^2}} = \frac{\pi}{2} \tag{3.22}$$

if the quartic polynomial coming out from the quantity in the radical has positive roots, u_1 and u_2. Equation (3.22) means that all bounded trajectories with $J \neq \mu$ are closed. For (3.21), the Hamiltonian (2.6) takes the form

$$H_\mu = \frac{1}{ar^2+b}\left(\frac{1}{2}\sum p_j^2 + \frac{\mu^2}{2r^2} + \frac{d\mu^2}{2}r^2 + \frac{c\mu^2}{2}\right) \tag{3.23}$$

which covers the MIC-harmonic oscillator Hamiltonian (2.18). We then call the Hamiltonian system $(T^*(\mathbb{R}^3 - \{0\}), \omega_\mu, H_\mu)$ with (3.23) the harmonic oscillator-type system, and hence the metric (2.2) with (3.21) the harmonic oscillator-type metric, denoting it by $\mathrm{d}s_{\mathrm{H}}^2$;

$$\mathrm{d}s_{\mathrm{H}}^2 = (ar^2+b)(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)) + \frac{(ar^2+b)r^2}{1+cr^2+\mathrm{d}r^4}(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)^2. \tag{3.24}$$

We have so far discussed closed trajectories in the case of $J \neq \mu$. Now we have to deal with the case of $J = \mu$, where the motion takes place in the direction of $\boldsymbol{J}$. For the Kepler-type system, we obtain from (3.2) the equation

$$\tfrac{1}{2}(a+br)^2\dot{r}^2 = (bE - \tfrac{1}{2}d\mu^2)r^2 + (aE - \tfrac{1}{2}c\mu^2)r - \tfrac{1}{2}\mu^2. \tag{3.25}$$

A comparison of this equation with the quantity in the radical of (3.18) shows that bounded motion will take place between $r_2 \leqslant r \leqslant r_1$ with $r_i = 1/u_i$, $i = 1, 2$. The trajectory is then closed, of course. For the harmonic oscillator-type system, the equation we have is written as

$$\tfrac{1}{2}(ar^2+b)^2r^2\dot{r}^2 = (aE - \tfrac{1}{2}d\mu^2)r^4 + (bE - \tfrac{1}{2}c\mu^2)r^2 - \tfrac{1}{2}\mu^2. \tag{3.26}$$

In view of (3.22) and (3.26), we see that the bounded motion takes place between $r_2 \leqslant r \leqslant r_1$ as well, where r_1 and r_2 are roots of the quartic polynomial of the right-hand side of (3.26).

Thus we have shown that all the bounded trajectories of both the Kepler-type system and the harmonic oscillator-type system are closed. The above discussion results in the following theorems:

Theorem 3.1. Among the Hamiltonian systems $(T^*(\mathbb{R}^3 - \{0\}), \omega_\mu, H_\mu)$, there exist two kinds of systems in which all of the bounded trajectories are closed; one is the Kepler-type system with the Hamiltonian (3.19) and the other is the harmonic oscillator-type system with the Hamiltonian (3.23).

Theorem 3.2. The Euclidean Taub–NUT metric is generalized to the Kepler-type metric and to the harmonic oscillator-type metric, given by (3.20) and (3.24), respectively, whose associated dynamical systems are the Kepler-type system and the harmonic oscillator-type system stated in theorem 3.1.

Closing this section, we remark on the MIC-harmonic oscillator. We have already shown that all the trajectories of the MIC-harmonic oscillator are closed. In fact, for $f(r)$ and $g(r)$ given in (2.17), equation (3.22) turns into

$$\Delta\phi = \int_{u_1}^{u_2} \frac{J\,\mathrm{d}u}{\sqrt{2E - ku^{-2} - J^2u^2}} = \frac{\pi}{2} \tag{3.27}$$

which is the same as those used for the ordinary harmonic oscillator, so that all the trajectories are closed for the MIC-harmonic oscillator as well.

4. First integrals and symmetry

In this section, for the Kepler-type system and the harmonic oscillator-type system, first integrals other than the angular momentum vector J will be found, and the Poisson brackets among them will be calculated.

We start with the Kepler-type system. It is well known that for the Taub–NUT metric, the vector

$$R_T = p \times J - 4m\left(E - \left(\frac{\mu}{4m}\right)^2\right)\frac{x}{r} \tag{4.1}$$

is a conserved vector, where E is the total energy (Gibbons and Manton 1986). In view of (4.1), we can assume that a vector similar to (4.1) is a conserved vector for the Kepler-type system;

$$R = p \times J - \alpha\frac{x}{r} \tag{4.2}$$

where α is a constant of motion. Then the condition that R is a conserved vector, $\mathrm{d}R/\mathrm{d}t = 0$, for the equation of motion (2.7) turns out to be

$$\alpha = -\frac{r^2 f'(r)}{2f(r)}|p|^2 - \mu^2\frac{r^2 g'(r) f(r)}{2g(r)^2} + \frac{\mu^2}{r}. \tag{4.3}$$

Since $f(r)$ and $g(r)$ for a Kepler-type system are given by (3.17), condition (4.3) is satisfied and results in

$$\alpha = aH_\mu - \frac{c\mu^2}{2} \tag{4.4}$$

where H_μ is the total energy (3.19). Thus we have obtained the Rung–Lenz-like vector R, which is a generalization of (4.1). We here notice that, though the Kepler-type system is derived in this article by applying Bertrand's method for closed trajectories, the system can be derived as well from the assumption of the conserved vector of the form (4.2). For the detail of the derivation, see Iwai and Katayama (1993).

The existence of the angular momentum vector J and the Runge–Lenz-like vector (4.2) with (4.4) shows that the trajectories are conic sections on the cone determined by (2.9). Trajectories are hyperbolae, parabolae, or ellipses, depending on whether $2bH_\mu - d\mu^2$ is a positive, zero or negative constant (see Iwai and Katayama (1993) for details). This fact is the same as in the case of the Taub–NUT metric (Gibbons and Manton 1986, Gibbons and Ruback 1987, 1988, Fehér and Horváthy 1987, Cordani, Fehér, and Horváthy 1988, 1990).

The commutation relations among the first integrals are

$$\begin{aligned}
\{J_\ell, J_k\} &= \sum \varepsilon_{\ell kj} J_j \\
\{J_\ell, R_k\} &= \sum \varepsilon_{\ell kj} R_j \\
\{R_\ell, R_k\} &= (\mathrm{d}\mu^2 - 2bH_\mu) \sum \varepsilon_{\ell kj} J_j
\end{aligned} \tag{4.5}$$

where J_ℓ and R_ℓ are the components of the conserved vectors J and R, respectively. The Poisson bracket is, of course, defined with respect to the symplectic form (2.5); for functions A and B, the Poisson bracket is given by

$$\{A, B\} = \sum \left(\frac{\partial A}{\partial x_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial x_i} \right) - \frac{\mu}{r^3} \sum \varepsilon_{ijk} x_i \frac{\partial A}{\partial p_j}\frac{\partial B}{\partial p_k}. \tag{4.6}$$

According to whether $d\mu^2 - 2bH_\mu$ is a positive, zero, or negative constant, the commutation relations (4.5) are those for the Lie algebra of $SO(4)$, $E(3)$, or $SO(1,3)$, respectively.

We now turn to the harmonic oscillator-type system. For setting up, we first mention the MIC-harmonic oscillator. For $f(r)$ and $g(r)$ given in (2.17), the equation of motion (2.7) can be put in the form

$$\frac{\mathrm{d}^2 \boldsymbol{x}}{\mathrm{d}t^2} = \frac{\mu}{r^3} J - k\boldsymbol{x} \tag{4.7}$$

where J is given by (2.8). From this, one has the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} J \times \boldsymbol{x} = -kJ \times \boldsymbol{x} \tag{4.8}$$

which means that $J \times \boldsymbol{x}$ is subject to the equation of motion for the ordinary harmonic oscillator. Using (4.8), McIntosh and Cisneros (1970) derived a dyadic constant of motion in the same manner as that for the harmonic oscillator. However, unfortunately, those constants of motion (i.e. the components of the dyadic) are not closed under the Poisson bracket to form the Lie algebra of $SU(3)$. For the quantized MIC-harmonic oscillator, a spectral generating algebra has been studied in Jakiew (1980), and renewed interest is found for accounting for the degeneracy of the energy levels (Labells, Mayrand, and Vinet 1991).

To deal with the symmetry for the harmonic oscillator-type system, it is of great use to introduce new variables (X_j, P_j) after Boulware, Brown, Cahn, Ellis, and Lee (1976). The transformation they defined, called the BBCEL transformation, is given by

$$\begin{aligned}
X &= \left(\boldsymbol{x} - \mu\frac{rJ}{|J|^2}\right)\left(1 - \frac{\mu^2}{|J|^2}\right)^{-1/2} \\
P &= \left(p - \mu\frac{(p \cdot \boldsymbol{x})J}{r|J|^2}\right)\left(1 - \frac{\mu^2}{|J|^2}\right)^{-1/2}
\end{aligned} \tag{4.9}$$

where $|J| > |\mu|$ has been assumed. Let us recall that, from (2.10), the condition $|J| > |\mu|$ is equivalent to $|J| \neq |\mu|$ and means that the cone on which trajectories lie does not contract to the axis in the direction of $\boldsymbol{J}$. Geometrically speaking, the BBCEL transformation in the case of $|\boldsymbol{J}| > |\mu|$ maps trajectories on the cone to those on the plane perpendicular to $\boldsymbol{J}$, since $\boldsymbol{X}\cdot\boldsymbol{J} = \boldsymbol{P}\cdot\boldsymbol{J} = 0$. A straightforward calculation shows that the BBCEL transformation satisfies

$$|\boldsymbol{X}| = |\boldsymbol{x}| = r \tag{4.10}$$

$$\boldsymbol{L} := \boldsymbol{X}\times\boldsymbol{P} = \boldsymbol{x}\times\boldsymbol{p} + \mu\frac{\boldsymbol{x}}{r} = \boldsymbol{J} \tag{4.11}$$

$$\boldsymbol{X}\cdot\boldsymbol{P} = \boldsymbol{x}\cdot\boldsymbol{p} \tag{4.12}$$

$$|\boldsymbol{P}|^2 = |\boldsymbol{p}|^2 + \frac{\mu^2}{r^2} \tag{4.13}$$

$$H_\mu = \frac{1}{2f(r)}\left(|\boldsymbol{P}|^2 - \frac{\mu^2}{r^2}\right) + \frac{\mu^2}{2g(r)}. \tag{4.14}$$

According to Fehér (1987), the standard symplectic form ω_B (defined below) is pulled back by the BBCEL transformation to the symplectic form ω_μ defined in (2.5),

$$\omega_B := \sum \mathrm{d}P_j \wedge \mathrm{d}X_j = \omega_\mu. \tag{4.15}$$

Thus the reduced system $(T^*(\mathbb{R}^3-\{0\}), \omega_\mu, H_\mu)$ with the restriction $|\boldsymbol{J}| > |\mu|$ is transformed into the dynamical system (M_B, ω_B, H_B), where M_B is the phase space given by

$$M_B = \{(\boldsymbol{X}, \boldsymbol{P});\ |\boldsymbol{X}\times\boldsymbol{P}| > |\mu|\} \tag{4.16}$$

and H_B is the Hamiltonian defined through (4.14),

$$H_B = \frac{1}{2f(r)}|\boldsymbol{P}|^2 - \frac{\mu^2}{2r^2 f(r)} + \frac{\mu^2}{2g(r)}. \tag{4.17}$$

Note that the BBCEL transformation (4.9) is invertible.

For the harmonic oscillator-type system, the Hamiltonian (4.17) can be rewritten as

$$H_B = \frac{1}{ar^2+b}\left(\tfrac{1}{2}|\boldsymbol{P}|^2 + \tfrac{1}{2}d\mu^2 r^2 + \tfrac{1}{2}c\mu^2\right). \tag{4.18}$$

We rewrite this equation as

$$bH_B - \tfrac{1}{2}c\mu^2 = \tfrac{1}{2}|\boldsymbol{P}|^2 + \tfrac{1}{2}(d\mu^2 - 2aH_B)r^2. \tag{4.19}$$

If we think of the H_B in the right-hand side as a constant, and if the coefficient $\mathrm{d}\mu^2 - 2aH_B$ is positive, then the right-hand side may be considered as the Hamiltonian for the ordinary harmonic oscillator, so that we may expect that the tensor defined by

$$M_{jk} := P_j P_k + (d\mu^2 - 2aH_B)X_j X_k \tag{4.20}$$

will be conserved. In fact, one can show by a straightforward calculation that

$$\{H_B, M_{jk}\}_B = 0 \tag{4.21}$$

where the Poisson bracket is defined through the symplectic form ω_B, which is the standard one. Note here that the Hamiltonian H_B in the right-hand side of (4.20) is treated as a function in calculating (4.21).

The Poisson brackets among first integrals, M_{ij} and $\boldsymbol{L} = (L_k) = \boldsymbol{X}\times\boldsymbol{P}$, are shown to satisfy

$$\begin{aligned}
\{L_i, L_j\}_B &= \sum \varepsilon_{ijk} L_k \\
\{M_{ij}, L_k\}_B &= \sum \varepsilon_{\ell ik} M_{\ell j} + \sum \varepsilon_{\ell jk} M_{i\ell} \\
\{M_{ij}, M_{k\ell}\}_B &= (d\mu^2 - 2aH_B)\sum(\delta_{ik}\varepsilon_{j\ell m} + \delta_{i\ell}\varepsilon_{jkm} + \delta_{jk}\varepsilon_{i\ell m} + \delta_{j\ell}\varepsilon_{ikm})L_m.
\end{aligned} \tag{4.22}$$

From this we conclude that if $d\mu^2 - 2aH_B$ is a positive constant, these first integrals form the Lie algebra of $SU(3)$. Thus we recognize that the harmonic oscillator-type system admits the $SU(3)$ symmetry if $d\mu^2 - 2aH_B > 0$. Note here that if $d\mu^2 - 2aH_B$ is a positive constant, so is $2bH_B - c\mu^2$ from (4.19), so that bounded trajectories must occur. Further, the total energy $E = H_B$ must satisfy the inequality $c\mu^2/2b < E < d\mu^2/2a$ if $a > 0$, $b > 0$, which makes a remarkable contrast with the ordinary harmonic oscillator.

We can treat the Kepler-type system as well by the use of the BBCEL transformation. In a similar manner to (4.19), we can obtain the expression for the Hamiltonian of this system as

$$bH_B - \frac{d\mu^2}{2} = \tfrac{1}{2}|\boldsymbol{P}|^2 - \left(aH_B - \frac{c\mu^2}{2}\right)\frac{1}{r} \tag{4.23}$$

which suggests that

$$\boldsymbol{R}_B = \boldsymbol{P} \times \boldsymbol{L} - \left(aH_B - \frac{c\mu^2}{2}\right)\frac{\boldsymbol{X}}{r} \tag{4.24}$$

may be a conserved vector. One can actually show that $\{H_B, \boldsymbol{R}_B\}_B = 0$. Further, the $\boldsymbol{R}_B$ is shown to be related to the Runge–Lenz-like vector $\boldsymbol{R}$ given by (4.2) with (4.4);

$$\boldsymbol{R}_B = \left(1 - \frac{\mu^2}{|\boldsymbol{J}|^2}\right)^{-1/2}\left(\boldsymbol{R} + \mu\left(aH_\mu - \frac{c\mu^2}{2}\right)\frac{\boldsymbol{J}}{|\boldsymbol{J}|^2}\right). \tag{4.25}$$

As is expected, the constants of motion $\boldsymbol{L}$ and $\boldsymbol{R}_B$ satisfy the same commutation relations as (4.5) under the Poisson brackets $\{\ ,\ \}_B$.

5. Concluding remarks

In this article, Bertrand's method played a central role. This method provides a guiding principle for finding dynamical systems of particular interest. We have indeed accomplished non-trivial generalization of the Kepler problem and the harmonic oscillator in association with generalized Taub–NUT metrics. Other dynamical systems were found after the same method about a decade ago; one of the authors (NK) and Ikeda (1982) applied Bertrand's method to a central-potential problem on a space of constant curvature to find a generalized Kepler problem and a generalized harmonic oscillator, which systems are the same as those found by Nishino (1972) by investigating first integrals quadratic in momenta.

Geometric properties of the Kepler-type metric have been studied in the preceding paper (Iwai and Katayama 1993), in which the Riemann and the Weyl curvature tensors of this metric are investigated from the view point of self-duality and conformal self-duality. On the other hand, the Kepler-type system can be considered as a generalization of the MIC–Kepler problem. The latter system has been studied extensively in a series of papers (Iwai and Uwano 1986, 1988, 1991a, b, Iwai 1993) from the view point of global symmetry. The global symmetry of the Kepler-type system and of the harmonic oscillator-type system will be a future problem to study.

References

Atiyah M F and Hitchin N 1985 *Phys. Lett.* **107A** 21–5
Bertrand J 1873 *C.R. Acad. Sci. Paris* 77 849–53
Boulware D G, Brown L S, Cahn R N, Ellis S D and Lee C 1976 *Phys. Rev.* **14** D 2708–27
Cordani B, Fehér Gy and Horváthy P A 1988 *Phys. Lett.* **201B** 481–86
——1990 *J. Math. Phys.* **31** 202–211

Fehér L Gy 1987 J. Math. Phys. **28** 234–9
Fehér L Gy and Horváthy P A 1987 *Phys. Lett.* **183B** 182–6
Gibbons G W and Manton N S 1986 *Nucl. Phys.* B **274** 183–224
Gibbons G W and Ruback P J 1987 *Phys. Lett.* **118B** 226–30
——1988 *Commun. Math. Phys.* **115** 267–300
Greenberg D F 1966 *Am. J. Phys.* **34** 1101–9
Ikeda M and Katayama N 1982 *Tensor, N.S.* **38** 37–40
Iwai T and Uwano Y 1986 *J. Math. Phys.* **27** 1523–9
——1988 *J. Phys. A: Math. Gen.* **21** 4083–104
——1991a *Nuovo Cimento* B **106** 849–71
——1991b *Nuovo Cimento* B **106** 1195–219
Iwai T 1993 *J. Phys. A: Math. Gen.* **26** 609-30
Iwai T and Katayama N 1993 *J. Geom. Phys.* **12** 55–75
Jakiew R 1980 *Ann. Phys., NY* **129** 183–200
Labelle S, Mayrand M and Vinet L 1991 *J. Math. Phys.* **32** 1516–21
Manton N S 1982 *Phys. Lett.* **110B** 54–6
——1985 *Phys. Lett.* **154B** 397–400
——1985 *Phys. Lett.* (E) 1985 **157B** 475
McIntosh H V and Cisneros A 1970 *J. Math. Phys.* **11** 896–916
Nishino Y 1972 *Math. Japan* **17** 59–67
Zwanziger D 1968 *Phys. Rev.* **176** 1480–8